RISK SHARING CONTRACTS

IN PROJECT APPRAISAL

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INTRODUCTION

Risk analysis techniques in the appraisal of capital investment projects have lagged behind both the theoretical developments and the actual conduct of project financing. While project appraisal techniques have seen significant advances in the analysis of projects from the various perspectives of the private investors, government, labor, consumers and the economy as a whole (see, for example, Jenkins and Harberger (1989)), this analysis has generally been conducted in a deterministic framework. The analysis of risk from these same various perspectives have received less attention. Moreover, the analysis of the important role of contracts to reduce risk and enhance the attractiveness of a project has received even less attention. This paper attempts to draw together advances in a number of areas of theory and practice within the framework of Monte Carlo simulation techniques to demonstrate how these techniques can be used to analyze the effects of various Types of contracts on the riskiness and attractiveness of projects to the various parties involved. The spirit of this paper is exploratory rather definitive. It is seeking to integrate developments from various sources to improve the analysis and management of risk in capital investment projects.

The major 'inspiration for this paper arises from the observation that, while risk-sharing contracts have come to play an important role in the practice of "project financing" for major mining, extraction, electrical power generation and other industrial projects, the techniques for designing and analyzing these contracts are inadequate. For example, in a mining project, a contract with a purchaser of the ore that guarantees some minimum sales volume and/or price changes *both the expected return and the riskiness of the project* from the perspectives of both the financiers and the purchaser. Another case arises where electricity generating facilities are being built, owned and operated in developing countries financed by foreign exchange loans. Financiers will seek electricity pricing agreements that are sensitive to exchange rate fluctuations and/or guarantees for the repayment foreign exchange loans. Risk analysis techniques should allow the evaluation of a range of such contracts, and by extension, assist in the choice of the "best" contract design. Simple techniques such as sensitivity or scenario analysis do not provide adequate measures of the changes in the return or the risk of the project from the perspectives of the various parties involved. Hence, they are not helpful in evaluating different risk-sharing contracts. Monte Carlo simulation techniques (MCSTS) for conducting risk analysis based on spreadsheet models of an investment project can be used to explore the effects of different contractual arrangements.[1] Computer software developments have made the conduct of risk analysis accessible and relatively simple within the usual limitations of data availability.[2] In

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[1] Monte Carlo simulation techniques have been used for over thirty years in the appraisal of capital investment projects. See for example, Brealey and Myers (1981), Chapter 10, for references to early work in this area.

[2] At least two programs are known to exist that allow the simple conversion of a deterministic spreadsheet model into a stochastic model in order to conduct risk analysis

addition, in recent years a greater consciousness has developed of the significant effects on estimates of expected values brought about by natural, legislated or contractual limits on the range of values over which observations or actual variables values are distributed compared to their potential or latent distributions. To deal with this, significant progress has been made in the estimation procedures for parameters based on such truncated or censored distributions.[3] Limiting conditions on prices and quantities form the essential features of many risk-sharing contracts. The MCSTs allow estimates of the changes arising from contractual arrangements in the mean, variance, and, in fact, the entire shape of the distribution of the value of an investment project from the perspectives of different parties.

The techniques that will be discussed in this paper parallel those used in contingent claims analysis (CCA) as discussed, for example, in Mason and Merton (1985), Hull (1989) or Dixit and Pindyck (1993). A few differences can be noted between CCA and the MCSTs that are discussed in this paper. CCA techniques have been used primarily in the valuation of derivative securities such as stock options, loan guarantees, or debt and equity under limited liability; whereas the MCSTS are used here to evaluate risk-sharing contracts operating in the product, labor or materials supply markets. Such contracts in the real' (as opposed to financial) markets change the expected distribution of the cash flows out of which the various debt and equity holders can be paid, and hence, the contracts alter their perceived risks and returns from investing in a project. CCA techniques can also be used to evaluate risk-sharing contracts or options on the real side of a project,[4] and MCSTs can also be used to value derivative securities. The central difference between these techniques, however, lies in the methods they use for estimating the value of a derivative security, or risk-sharing, contract, Both techniques require a description of the nature of the process generating the distribution of values of the derivative or underlying security or cash flow. This is normally described by a differential equation or set of differential equations. CCA has sought analytical solutions to the market value of a contract (such as the Black-Scholes formula for estimating the value of a call option on a stock), or for more complex cases, numerical solution techniques are required; whereas the MCSTS produce

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"RiskMaster for windows' (by Savvalds C. Savvides, Nicosia, Cyprus and Master Solutions, Cambridge, Massachusetts) is a spreadsheet template model that can be attached to any spreadsheet model to generate and analyze Monte Carlo simulations."@RISK' (by Palisade Corporation, Newfield, New York) is a spreadsheet 'add-in' program that performs similar simulation and analytical functions.

[3] The estimation formula for the moments of some types of truncated distributions is not new. See Johnson and Kotz (1970). The development of estimation techniques for economic parameters based on truncated or censored data is more recent. See Maddala (1983) for the estimation of such "limited dependent variables."

[4] Mason and Merton (1985), for example, discuss the use of CCA to value the option of a project being able to switch production techniques, input suppliers or product lines depending upon market conditions, or to value the flexibility to suspend production if revenues did not cover variable costs.

an estimate of the entire distribution of the values of the derivative or underlying security or cash flow upon which risk-sharing contractual conditions are imposed. The values of the conditional distributions to the contractual parties can then be calculated. Where analytical solutions can be found and estimated, they obviously provide a quick and direct route to the solution. CCA techniques rely on observations from well functioning markets. MCSTS can be used in the evaluation risks and returns where markets for pricing risk are not well developed, or the focus is more on the analysis of the effects of changing than risk sharing arrangements. Risk-sharing contracts and the nature of the underlying distributions of values may be complex. MCSTs can readily be adapted to a spreadsheet model that contains any, possibly complex, set of contractual conditions imposed on any type of distribution of values. CCA typically uses a more constrained set of analytically tractable types of distribution for risk variables. In addition to this flexibility, the effects of a contract on values of an investment for the various parties involved can be estimated simultaneously.

The remainder of this paper discusses the role of risk-sharing contracts, presents the outline of the analysis of how risk-sharing contracts affect the returns and risks seen by the various contracting parties, and then discusses how some different contractual structures can be analyzed using MCSTS. Finally, a brief summary and conclusion section is provided. Technical appendices are provided that deal with the estimation of means and variances of truncated and censored normal and lognormal distributions. Based on these results, a general solution of the value of a call option on an equity share is derived as well as the specific Black and Scholes solution. This illustrates the parallels between the CCA and MCST approaches.

ROLE OF RISK-SHARING CONTRACTS

In a well-functioning competitive capital market, it is generally sufficient to assume that the costs of risk implicit in any capital investment project are minimized through the efficient risk diversification. Given a sufficiently large number of project owners each with a well diversified portfolio, the random or unsystematic elements of risk are effectively eliminated leaving only systematic risk to be compensated by higher (or 'adjusted') returns for an investment project. If this was a sufficient characterization of a capital investment project, risk analysis and management would reduce to estimating the magnitude of the systematic risk involved in the project and ensuring that conditions for well diversified ownership prevailed. In reality, a number of imperfections often enter them picture, especially with major capital investment projects such as large mining projects, which are likely to leave large elements of undiversified risk that require closer analysis and management.

A number of conditions may generate the imperfections in the efficient risk allocation by means of the capital markets. The large size of the required capital investment can generate relatively large lumpiness in ownership and, hence, incomplete elimination of the random element of the risk. Information problems about the availability of resources, about productivity or costs of new technologies, or about the nature of markets can be severe. Information problems also arise in predicting and monitoring the performance of the various participants in the project. Monopolistic or monopsonistic elements in product or raw material markets can create uncertainties for financiers. (Such uncompetitive market conditions, however, also create risk-

sharing opportunities.) Institutional constraints arising from the Political, regulatory or legal arenas can also hinder the efficient operation and financing of projects. Laws and regulations, for example, play critical roles in natural resource and utility markets. Hence, the efficient reduction of risk may not be feasible merely through the financial markets. Risk reduction may require risk-sharing contracts 'with purchasers, suppliers, labor and/or the general taxpayer. Such contracts form an alternative avenue for diversifying the risk, and changing the nature of the risk and return faced by project financiers.

It is also recognized that risk-sharing contracts overlap with the incentives problems that arise where full information about the behavior of the project participants is not possible - the traditional "principal-agent' incentives problem. Some contracts both change the distribution of the burden of risk and the incentives and, hence, also change the distribution of project outcomes. For example, workers who receive part of their wages based on the profit performance of a business not only share in the risk of the project, but also face a different set of work incentives. This paper only focuses on the risk distribution issue and ignores the behavioral effects of changing incentives. The MCSTs discussed here in no way limit the inclusion of estimates of behavioral changes arising out of different contracts.

Two major types of risk sharing contracts can be identified. *The first type limits the range of values of a cash f7ow*: it either puts a *lower bound on sales revenues,* or it puts an *upper bound on costs.* On the sales revenue side, a purchaser, particularly one concerned by the security of supply of some commodity, for example, may be willing to agree to purchase a minimum quantity and/or a at minimum price. Alternatively, the purchaser may agree to a limited price range, or a fixed price growth path, or long-run average price.[5] Such contracts (which include so-called "take and pay" contracts) reduce the uncertainty in the cash inflows. This both increases the expected return on the project and provides the financiers with greater security in financing the project. A more extreme guarantee of sales revenue is where a purchaser guarantees a stream of minimum payments to the project irrespective of the price actually paid and quantity of the product actually acquired by the purchaser. Such contracts are sometimes referred to as 'take or pay' contracts. Without some such sales contract guaranteeing some level of cash inflow, many large projects would have difficulty in raising the necessary capital funds. In small poor countries, even the government may not have the fiscal capacity to provide a credible guarantee on the loan repayments of large capital intensive products. These types of contract have become common in the 'limited recourse' project financing arrangements for large mining and electrical power production projects.

On the materials cost side, a supplier may be willing to guarantee some minimum quantity and/or some maximum price to the project. Alternatively the supplier may be willingly to agree to a limited price range, or fixed price growth path, or long-run average price. Again this type of

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[5] Where options, futures and forward markets exist for a commodity being produced by a project, these can be used as a way of reducing the variability in prices of the project sales. Such markets, however, are not generally available for all commodities, and the time period over which contracts are available in most formal secondary markets are often too short to remove enough of the revenue uncertainty to be substitutes for long-term contracts with identified purchasers.

contract increases the returns and/or decreases the variability in the returns to the project financiers. Contracts on each cost item obviously are individually less important dun a sales contract. Taken collectively, however, their importance rises. In addition, if the feasibility of a project is dependent on some special input with restricted sources of supply, then contracts guaranteeing the supply of this input may be critical.

The second type of risk-sharing contract reduces the risk borne by financiers by increasing the correlation between some cost item and the sales revenue. This decreases the variance in the return to the financiers. Examples of such contracts include bonds with interest rates indexed to the product sales price, or labor contracts that base a share of wages on the project sales performance or profits.

In the appraisal of investment projects, other options may be also available to project managers to reduce risk and increase returns. Mason and Merton (1985) note the importance of taking into account the value of the option to temporarily discontinue production when revenues cannot cover variable costs or the option to switch technologies, inputs or product lines in response to changing relative prices. The analysis of the value of the flexibility to avoid extreme costs or take advantage of new market opportunities can also be undertaken using either contingent claims analysis techniques or MCSTS. Dixit and Pindyck (1993) provide extensive analytical development of the evaluation of the benefits of delaying irreversible investments to gain information about risky product, raw material or investment prices, or about new technology costs or productivity. They provide parallel analysis of value of the option to delay or restructure an investment using CCA approaches and using traditional discounted cash-flow analysis as employed in the MCSTs discussed here.

ANALYSIS OF RISK-SHARING CONTRACTS

In the analysis of the riskiness of projects due to unsystematic risk, Monte Carlo simulation techniques (MCSTS) have long provided the basic methodology. See Pouliquen (1970), Reutlinger (1970) and Savvides (1994). This approach has emphasized the identification of the major risk variables and the specification of their distribution in the context of appraising the attractiveness of investment projects based on projected incremental cash flows. In addition, the identification of all the 'within project' correlations among the risk variables has been stressed in order to have the correct model of the project cash flows. A positive systematic relationship or correlation between a revenue source and a cost item, for example, decreases the predicted variance in the bottom-line net cash flows; whereas a positive correlation between two revenue items (or two cost items) increases the predicted variance in the net returns. Such MCSTs are then used to produce the distribution of outcomes - net present values (NPVS) - of the project to assist the investor in deciding upon the attractiveness of the project. In summary, the application of the MCST in investment appraisal is analogous to any forecasting model. Forecasting models are based on the estimated -systematic relationships between the predicted outcome and explanatory variables plus a random or unsystematic residual component following some probability distribution. A model of the predicted incremental cash flows, similarly, has to be developed that contains all the systematic relationships between the component cash flows as

well as identifies all the unexplained or residual variance in these components.[6] The MCST essentially provides a way of "adding up" the residual errors in the component cash flows of the model to ascertain their impact on the possible distribution of the NPVs from the perspectives of the investors as debt or equity holders, or even the general taxpayer in the context of an economic appraisal of the project.

The incorporation of risk-sharing contracts into MCSTs requires the simple addition to the model of the sets of contractual conditions attached to specific cash flows. This is a simple task in an electronic spreadsheet-based cash flow model. The consequences of the contract can then be ascertained by simulating the project with and without the contractual limitations. The changes in the expected values and distributions of selected cash flows experienced by different parties - equity holders, debt holders, government, labor, product purchasers or materials suppliers - can be calculated. A more sophisticated analysis would also build expected behavioral responses to changing price incentives into the model.

The effects of the two basic types of contract on the returns and risks faced by the various parties involved in a contract can be presented somewhat more formally. A contract that puts limits on the range of prices or quantities at which a good will be traded typically results in the price or quantity having a censored distribution. For example, where a purchaser guarantees a floor price (p_B.), then, if the market price falls below this price, the trade occurs at p_B; otherwise, if the market price exceeds p_B the trade occurs at the market price. This results in the seller receiving a higher price on average, the buyer receiving a correspondingly lower price on average, and both parties facing, a lower variability in the price than would have existed without the contract. See Figure 1. Now, if μ would have been the average price without the contract guaranteeing the floor price of p_B, then μ can be expressed in terms of the means of the two parts of the overall price distribution, namely, μ_A, the mean of the prices below p_B, and μ_B, the mean of the prices above P_B, as follows:

(1) $$\mu = \text{Prob } (p \leq \underline{pB}) \cdot \mu_A + \text{Prob}(p > p_B) \cdot \mu_B$$

The overall distribution of the price can be viewed as consisting of two truncated distributions: one truncated from above at p_B with mean μ_A and composed of a proportion [Prob $(p \leq p_B)$] of the overall price distribution, and the other truncated from below at p_B with mean μ_B and composed of a proportion [Prob $(p > p_B)$] of the overall price distribution. See Figure 1, panels B1 and B2. (Appendices A and B give the relationships between the means of truncated normal and lognormal distributions and the parameters of their underlying distributions.) In the case of the example of a guaranteed floor price, all values of the distribution truncated from above are concentrated at p_B. The overall distribution is now referred to as a censored

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[6] For example, a petrochemical plant using oil-based feedstocks would have a positive correlation between the petrochemical product price and the price of the feedstock. This could be modeled either by estimating a correlation coefficient between their prices, or by estimating models that relate their prices to common determinants such as world oil prices and exchange rates.

distribution and μ_A in (1) is replaced by p_B. See Figure 1, panel C. Hence, the mean for the censored distribution (μ_B^*) becomes:

(2) $\qquad \mu_B^* = \text{Prob } (p \le p_B). \; p_B \; + \; \text{Prob } (p > p_B). \; \mu_B$

Clearly μ_B^* exceeds μ_B. In addition, the variance of the censored distribution will be less than the variance of the overall price distribution as all prices below p_B are now concentrated at p_B. This implies that the project revenues and, consequently, the expected NPV of the project will have a higher expected value and lower variance. This clearly increases the attractiveness of the project to the financiers. The purchaser of the project output, however, faces an increased expected price, but can trade this off against lower price variability and possibly an increased security of supply.

Explicit expressions can be derived for the mean and variance of some common truncated and censored distributions such as for normal or log normal distributions. See Appendices A and B. Using MCSTS, however, as long as the overall distribution can be specified, the mean and variance of the overall and of any censored or truncated distribution based on the overall underlying distribution can always be calculated even if a nonstandard type of distribution is used.

The other major type of risk-sharing arrangement discussed above relies on exactly the same fundamental considerations that underlie risk reduction through portfolio diversification. The variance of any combination of returns or cash flows depends upon their individual variances plus the covariance between them. In general, for any two returns, X and Y, with variances $V(X)$ and $V(Y)$, the combined variance is given by:

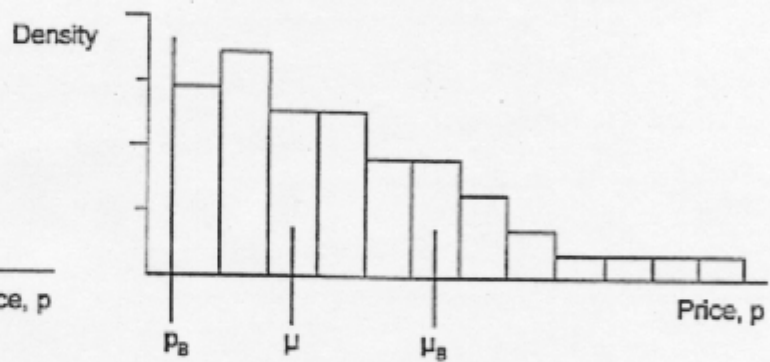Figure 1: Truncated and censored distributions and their mean values

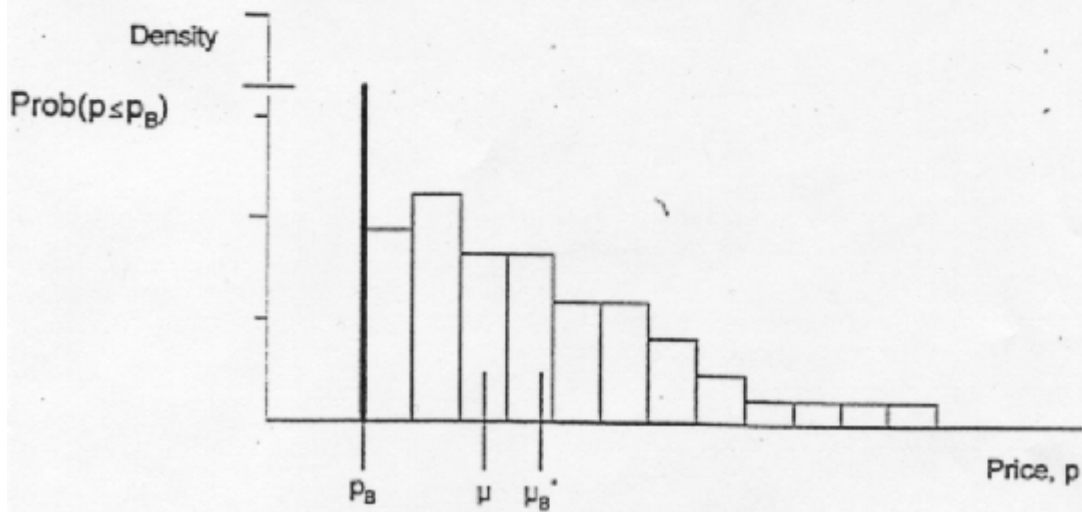A. Overall or latent price distribution

Density

p_B μ
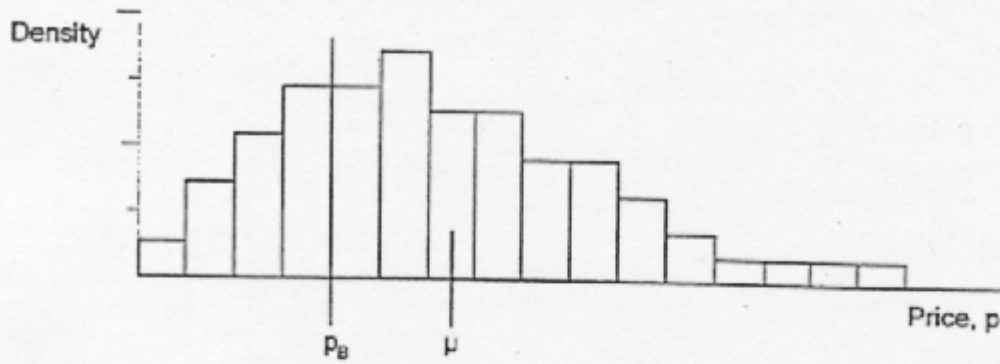
Price, p

B1. Price distribution truncated
from above at p_B

Density

Price, p

μ_A p_B
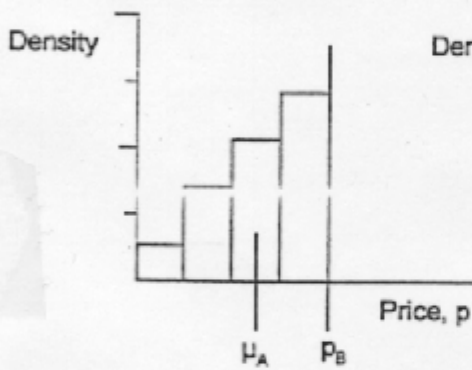
B2. Price distribution truncated
from below at p_B

Density

Price, p

p_B μ μ_B

C. Price distribution censored at p_B

Density

Prob($p \leq p_B$)

p_B μ μ_B^*

Price, p

(3) $(X + Y) = V(X) + V(Y) + 2COV(X,Y)$

and, if these returns are subject to any scale factors (constants a or b), then

(4) $V(aX + bY) = a^2V(X) + b^2V(Y) + 2abCOV(X,Y).$

In the context of portfolio diversification, if Y is an incremental return being added to an existing portfolio with return X, then, from (3), its incremental contribution to the overall portfolio variance is lower, the lower its own variance, $V(Y)$, and the lower its covariance with existing portfolio returns, $COV(X,Y)$. The benefits of diversification can also be readily seen from (4). If instead of all investments being made in X, a share is also put in Y, then, assuming an equal share in each investment $(a = b = \frac{1}{2})$, the combined portfolio variance becomes:

(5) $V(\frac{1}{2} X + \frac{1}{2} Y) = \frac{1}{4} V(X) + \frac{1}{4} V(Y) + \frac{1}{2} COV(X,Y)$

The variance of this combined portfolio will be less than $V(X)$ for sure, if $V(Y) \leq V(X)$ and $COV(X,Y) < V(X)$. More generally, the portfolio variance will be less than $\{aV(X)+bV(Y)\}$ as long as X and Y are not perfectly correlated.

In the context of risk-sharing contracts, the variance in the profits or returns of the equity holders from a project can be described in terms of (5) by setting $X = R$ (Revenues), $Y = C$ (Costs), $a = 1$, and $b = -1$, or profits = R - C. - Hence,

(6) $V(R - C) = V(R) + V(C) - 2COV(R,C)$

Now, by setting up contracts with suppliers of goods or services to the project that increase $COV(R,C)$, the variance in the profits can be lowered as long as in the process $V(C)$ is not raised by more dun twice the increase in $COV(R,C)$. Two such risk-sharing contracts include: indexing interest payments to the product price, and paying labor based on some project performance criteria such as profits.[7]

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[7] To illustrate the effect of a profit sharing contract, assuming that wages are the only costs, and that labor receives a share (x) of the wages they would have otherwise received plus a share (y) of the profit after this wage payment, then C becomes $\{xC + y(R - xC)\}$ and profits (R.-.C) become $\{(1-y)R - x(1-y)C\}$. It is clear that the new labor costs are now more positively related to R. The variance in the profits now becomes:

$$(1-y)^2V(R) + x^2(I-y)^2V(C) - 2x(l-y)^2COV(R,C)$$

Given $0 < x,y < 1$, it is clear that the variance in profits declines compared to the case where x = 1 and y =0.

APPLICATION OF MOINTE, CARLO SBIULATION TECBNIQUE

At the beginning, of the previous section, the broad outline was given of how MCSTs are used to analyze risk sharing contracts. Essentially a base case model of the investment decision has to be simulated. The NPV and other results of this have to be compared with simulation of the model adjusted to include the risk sharing contract (or similarly any other change in the investment decision such as delaying an investment to check whether expected gains from added flexibility to respond to new market price or technical information will be positive.) While the previous section, discussed briefly some of the theoretical basis for the gains from risk sharing contracts, this section focused on how these contracts are entered into an electronic spreadsheet analysis using MCSTS.[8]

In a Monte Carlo simulation of an investment decision model, the model is simulated repeatedly to generate the distributions of cash flows and NPVS. In each simulation, the variables subject to uncertainty are sampled from the density functions specified subject to any correlations specified between these variables. Risk sharing contracts enter the model by imposing conditions on the variable values selected on each simulation. For example, if a floor price is guaranteed on the product price, then a conditional function is placed on the price value that overrides the randomly selected price if it falls below the floor Price. This floor price may be constant or determined by some relationship that varies the price over time. Alternatively, if minimum revenue is guaranteed then a conditional function is placed on the revenue value that overrides the value derived from the randomly selected price and quantity values. On average such floor prices or revenues eliminate any low revenue cases and raise the expected NPV to equity, but lower its variance. The probability of default on debt would decline, while the expected tax payments would increase. At the same time, any minimum product quantity guarantee (as in a take-and-pay contract) can be expected to raise the usage of variable inputs which can have uncertain effects on the NPV to equity and tax revenues unless there is also some floor price or revenue guarantee. A minimum revenue guarantee, however, without a minimum quantity guarantee (as in a take-or-pay contract) does not have the same impact on variable costs - the revenues can be earned in some circumstances without incurring the costs. This is clearly highly favorable to the financiers and the tax collectors.

The second basic type of contract concerned increasing the positive correlations between the revenue and cost streams in order to decrease the variance in returns to equity and also raise the expected returns to the extent that downside losses can be avoided. In some sectors, the correlation between revenues and major costs may be high. For example, a petrochemical

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[8] See Glenday (1989) for some numerical illustrations of simple contract simulations. The contract cases discussed exclude some generic "truncation" situations that are common in multi-year projects. First, a check is usually included in a spreadsheet model to raise short- term financing in years when net cash flow to equity becomes negative to recognize the liquidity constraint on operations. Second, a possible stoppage of production could be allowed in periods when revenues do not cover variable costs. Third, the effects of limited liability on equity holders on the NPVs to equity and total private capital have to be accounted for.

product primarily based on an oil-derived feedstock would display high correlation between the prices of the petrochemical product and the feedstock. The prices of both would be similarly affected by world oil prices and the exchange rate. By contrast, oil burning electrical power generation plant would have less intrinsic relationship between revenues and costs. The electricity price would be most likely regulated and set in local currency, while the oil costs would-fluctuate with world prices and the exchange rate. In addition, the plant may have been financed by foreign currency loans. Such an investment is highly risky for foreign financiers. They may require the indexation of the electricity price to oil and/or exchange fluctuations to rninimize the risk of project default as well as seeking guarantees against country risk -on the loan repayments. In such a case, the electricity price in the model becomes a function of the oil and/or foreign exchange rate, which are each, in turn, determined by their values selected from their estimated distributions. The indexing relationship may run the other way, for example, if a government was attempting, to stabilize the incomes of plantation owners facing high financing costs on a tree crop with a long pay- back period. In this case, the interest rate on loan finance could be indexed to the crop price. In the model, the interest rate then depends in the crop price, which is determined by its estimated probability distribution.

The other important broad class of contracts that increase correlation of revenues and costs are performance- or incentive-based labor contracts. These can cover a wide range of contracts: piece work (workers are only paid for what they produce); bonuses based on beating quantity targets for production or sales, or beating unit cost reduction targets; profit sharing agreements; stock options of various kinds; or share ownership by the workers. In each case, the time-based pay of workers would be somewhat reduced and replaced by some formula based pay. This formula has to be included in the model for its effects on the various NPV measures to be simulated. Generally, such contracts aim at transferring some of the investment risk from capital owners to labor, while at the same time improving labor productivity through the incentives provided to labor to share in the gains and reduce the losses. Equity holders seek both lower variance in their returns and higher returns. MCSTs can be used to determine the changes in labor and equity returns arising from the labor contract as well as all the other unrelated factors causing variations in the project revenue and cost cash flows.

SUMMARY AND CONCLUSIONS

The need to improve the analysis of changes in project returns and risks -caused by "contingent claims' or risk-reducing contracts is identified. Improved methods for the analysis of project risk not only would provide better estimates of project outcomes - the expected project net present values to the various parties involved in the investment as well as the distribution of these values - but it would also allow the analysis of different contractual structures. The latter capability is important in risk management or 'doing something' to reduce and/or efficiently redistribute the burden of risk and, as such, should form an essential element in the design of risky projects. Without some form of risk sharing contract, many large scale, but risky investments would not be undertaken. Monte Carlo simulation techniques provide a tractable framework for designing and analyzing risk sharing contracts.

APPENDIX

A. **Truncated and Censored Normal Distributions**

Notation

f (x) is the density function of a normal distribution and F(x) is the cumulative normal distribution with mean, μ, and standard deviation, σ.

$\phi(e)$ is a unit normal density function and Φ is a unit cumulative normal distribution with zero mean and unit standard deviation, and where $e = (x-\mu) /\sigma$.

In the context of contingent claims analysis, f(x) or $\phi(e)$ is the distribution of the value of the reference security that determines the value of the derivative security which typically would have a truncated or censored distribution derived from the distribution of the reference security. in the context of limited dependent variable estimation, f(x) or $\phi(e)$ is referred to as the latent distribution, while the related truncated and censored distribution is the observed distribution.

A.1. **Doubly Truncated Normal Distribution**

If the distribution of x is truncated from above at A ($x \leq A$) and from below at B ($x > B$, and $B < x \leq A$), then 100% of the distribution lies in the range from B through A and following Johnson and Kotz (1970: 81-85) has

i. a density function:

$$g(x) = \frac{\phi(e) / \sigma}{\Phi (d_A) - \Phi (d_B)}$$

where: $d_A = (A - \mu) /\sigma$ and $d_B = (B - \mu) / \sigma$

ii. a mean value:

$$\mu_{AB} = \mu + \frac{\phi (d_B) - \phi (d_A)}{\Phi (d_A) - \Phi (d_B)} \sigma$$

iii. a variance:

$$\sigma_{AB}^2 = \sigma^2 [1 + \frac{d_B \phi (d_B) - d_A \phi (d_A)}{\Phi (d_A) - \Phi (d_B)} - \{ \frac{\phi (d_B) - \phi (d_A)}{\Phi (d_A) - \Phi (d_B)} \}^2]$$

A.2.   Singly Truncated Normal Distributions

A.2.1.   Truncation From below

If x > B and A becomes infinitely large, then from A.1. above, the truncated distribution has
i.   a density function:

$$g(x) = \frac{\phi(e) / \sigma}{1 - \Phi(d_B)}$$

ii.   a mean value:

$$\mu_B = \mu + \frac{\phi(d_B)}{1 - \Phi(d_B)} \sigma > \mu$$

iii.   a variance:

$$\sigma_B^2 = \sigma^2 \left[1 + \frac{d_B \phi(d_B))}{1 - \Phi(d_B)} - \left\{ \frac{\phi(d_B)}{1 - \Phi(d_B)} \right\}^2 \right]$$

A.2.2.   Truncated from above
If x ≤ A and B becomes an infinitely large negative number, then from A.1. above, the truncated
distribution has
i.   a density function:

$$g(x) = \frac{\phi(e) / \sigma}{\Phi(d_B)}$$

ii.   a mean value:

$$\mu_A = \mu - \frac{\phi(d_A) / \sigma}{\Phi(d_A)}$$

iii.   a variance:

$$\sigma_A^2 = \sigma^2 \left[1 - \frac{d_A \phi(d_A)}{\Phi(d_A)} - \left\{ \frac{- \phi(d_A)}{\Phi(d_A)} \right\}^2 \right]$$

A.3.   Singly Censored Normal Distributions[9]
A.3.1.   Censored from below
If x ≤ B, then x = B, and if A becomes infinitely large, then from A.2.1. above, the censored distribution
has
i.   a density function:

$$h(x) = \left\{ \begin{array}{ll} \Phi(d_B) & \text{if } x \leq B \\ \phi(e) / \sigma = f(x) & \text{if } x > B \end{array} \right.$$

ii.   a mean value:

$$\mu_{B*} = \Phi(d_B)B + [1 - \Phi(d_B)] \left[\mu + \frac{\phi(d_B)}{1 - \phi(d_B)} \sigma \right] > \mu$$

and

$$\mu < U_{B*} < U$$

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[9] For the difference between truncated and censored distributions, see Maddala (1983: 149-151).

A.3.2.   Censored from above

If $x > A$, then $x = A$, and if B becomes an infinitely large -negative number, then from A.2.2. above, the truncated distribution has

i.   a density function:

$$h(x) = \begin{cases} \phi(e)/\sigma = f(x) & \text{if } x \le A \\ 1 - \Phi(d_A) & \text{if } x > A \end{cases}$$

ii.   a mean value:

$$\mu_A^* = \Phi(d_A)\left[\mu - \frac{\phi(d_A)}{\Phi(dA)}\sigma\right] + [1 - \Phi(d_A)]A < \mu$$

and $\mu_A < \mu_A^* < \mu$

B.   Lognormal distribution

B.1   Mean and variance of lognormally distributed variable

A variable y is lognormally distributed if $x = \ln(y)$ is normally is normally distributed.[10] If x has a mean, μ, and standard deviation, σ, then y has:

i.   Mean

$$\mu_y = \exp(\mu + \tfrac{1}{2}\sigma^2)$$

hence, $\mu_y > \exp(\mu)$

or   arithmetic mean of y (μ_y) > geometric mean of y [$\exp(\mu)$]

ii.   Variance

$$\sigma_y^2 = \exp(2\mu + \sigma)[\exp(\sigma^2) - 1]$$

B.2.   Lognormal distribution truncated from below

If $y > B$ [or $x > \ln(B)$], then, following Johnson and Kotz (1970: 129), the truncated lognormal distribution has a mean value

(B.2) $\mu_{yB} = \mu_y[1 - \Phi(L_B - \sigma)]/[1 - \Phi(L_B)]$

where $L_B = [\ln(B) - \mu]/\sigma$

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[10] See Johnson and Kotz (1970:112-3)

C. Black and Scholes Pricing Formula for Stock Option

The price of a European call option can be specified in terms of a censored distribution. If the current value of a stock is S_O, and on the exercise date T years hence, it is S_T, and if the exercise price is X, then, using A.3.1. above, the net present value of the call option (c) is given by[11]:

(C.1)
$$c = [\{(Prob\ (S_T \leq X)\ .X + Prob\ (S_T > X)\ .\ E\ (S_T\colon S_T > X)\} - X]I\ e^{-rT}$$
$$= Prob\ (S_T > X)\ .\ [E\ (S_T\colon S_T > X) - X]\ e^{-rT}$$
where r = risk free discount rate.

Now, if the stock price is expected to grow according to the following differential equation:

(C.2)
$$dS/S = (r - \sigma^2/2)dt + \sigma\varepsilon(dt)^4$$

where $\sigma$the variance in the annual is return of $(r - \sigma^2/2)$ and ε is a standard unit normal random error term,

then the distribution of ST is lognormal such that'2[12]
$$mean\quad ln\ (S_T)\qquad ln\ (S_O) + (r - \sigma^2/2)dt$$

and variance in $\quad ln\ (S_T)\ =\ \sigma^2 T.$

Hence, following (B.2) above

$$Prob\ (S_T > X)\quad =\quad 1\ -\ \Phi(L_B)$$

where: $L_B = [ln(X)\ -\ mean\ ln\ (S_T)]\ /\ std.\ Dev.\ of\ \ ln(S_T),$

and $\quad E\ (S_T\colon S_T > X)\quad =\quad S_o e^{rT}\quad \dfrac{1\ -\ \Phi(L_B\ -\ \sigma T)}{1\ -\ \Phi(L_B)}$

Hence,
$$c = S_o\ [1 - \Phi(L_B\ -\ \sigma T)]\ -\ Xe^{-rT}\Phi\ (D\ (-L_B)$$

which is the Black-Scholes analytical solution to the price of a call option on a stock assuming the growth in the stock price follows (C.2).

In general, the general expression for the value of the call option given by (C.1) can be solved using Monte Carlo simulation methods for any given stochastic differential equation such as (C.2) that gives the growth path of the stock value. This is achieved by simulating a distribution of S_T values by repeatedly solving the following:

$$S_T = S_O + \sum_{i=1}^{T} dS_i\ (\varepsilon_i)$$

for different sets of T 'values of r&, drawn at random from a unit normal distribution. dSi could be given by (C.2) or any other similar differential equation that is hypothesized to give the growth path in stock values. Once the distribution of ST has been established, the price of the option given by
(C. 1) can be found.

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[11] Note that the structure of this expression follows that of the mean value of a distribution censored from below given in A.3.1 with B = X, but without the distribution necessarily being normal. In addition, X, the exercise price, is deducted from the expected value and the resultant value is discounted to give a present value by multiplying it by e^{-rT}.

[12] $dS/S\ =\ dInS\ =\ lnS_T\ -\ lnS_O$

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